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Agreement Percolation and Phase Coexistence in Some Gibbs Systems

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We extend some relations between percolation and the dependence of Gibbs states on boundary conditions known for Ising ferromagnets to other systems and investigate their general validity: percolation is defined in terms of the agreement of a configuration with one of the ground states of the system. This extension is studied via examples and counterexamples, including the antiferromagnetic Ising and hard-core models on bipartite lattices, Potts models, and many-layered Ising and continuum Widom–Rowlinson models. In particular our results on the hard square lattice model make rigorous observations made by Hu and Mak on the basis of computer simulations. Moreover, we observe that the (naturally defined) clusters of the Widom–Rowlinson model play (for the WR model itself) the same role that the clusters of the Fortuin–Kasteleyn measure play for the ferromagnetic Potts models. The phase transition and percolation in this system can be mapped into the corresponding liquid–vapor transition of a one-component fluid.

KEY WORDS: Percolation; Gibbs measures; nonuniqueness; antiferromagnets; hard-core models; Widom–Rowlinson continuum model.

1. INTRODUCTION

The low-temperature phases of matter can generally be thought of as small thermal perturbations of corresponding ground states. This is particularly simple for the case of a classical lattice system whose configuration is specified by $\{\sigma(x)\}$ with x on some regular lattice $\mathscr{L}$, and $\sigma(x)$ a spin

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variable taking one of a finite number of values for each x. Given a local interaction having a finite number (greater than one) of periodic ground-state configurations (PGSC), we can then take these PGSC as boundary conditions for the Gibbs measures in Λ, i.e., consider the Gibbs distribution at inverse temperature β with boundary conditions given by a PGSC on Λ^c, the complement of some finite box Λ (see Section 2 for precise definitions). For fixed Λ and $\beta > 0$ the boundary condition has a definite influence on the probability distribution of the spins in the bulk of Λ. In particular, in the limit $\beta \nearrow \infty$, the Gibbs measure in Λ becomes concentrated on the extension inside Λ of the PGSC imposed outside Λ. We are interested in knowing whether this influence persists for β large but finite when we take the volume Λ to be macroscopic. In other words, is there an ordered state in the infinite-volume ($\Lambda \nearrow \mathscr{L}$) system depending upon the PGSC imposed as boundary conditions on the finite box Λ?

If there is such a *memory* of the state with respect to boundary conditions *at infinity*, one sometimes calls the corresponding ground state thermally stable. The well-known Peierls argument provides such a result for the ferromagnetic Ising model and the Pirogov–Sinai theory[27] and its extensions study the genericity of this scenario. Under certain conditions, e.g., when there is a finite number of thermally stable periodic ground states, Pirogov–Sinai theory allows one to construct the low temperature phase diagram of the system (for review see Sinai,[32] Zahradník,[36] Bricmont and Slawny,[8] and Slawny[33]).

In this paper we investigate the geometric or percolation picture of this memory effect, considered by many authors,[30,13,16] to make precise the intuition that the influence of the boundary conditions must propagate via "interacting sites" from infinity to the center of the system in order to be relevant there. We are thus led to the question: in what sense is the finite-temperature state corresponding to a ground state accompanied by the presence of an infinite connected cluster on which the PGSC is realized? This question is fully answered at sufficiently low temperatures where the proof of the existence of different phases, determined by different PGSC boundary conditions, via the Peierls argument or Pirogov–Sinai theory actually proves that in each such phase there is such a cluster and there is no infinite cluster on which another PGSC is realized. Our interest here is therefore primarily the extension of well-known low-temperature results to higher temperatures where the Peierls and Pirogov–Sinai arguments fail. We are particularly interested in the question of whether, for two-dimensional systems, the existence of an ordered state at a finite temperature which corresponds to a (thermal) perturbation of a ground state is equivalent to the percolation in this state of that and only that ground-state configuration. This is known, for example, in the case of the standard ferromagnetic Ising model without external magnetic field. As we show below, this picture extends to other models such as the antiferromagnetic Ising model, the hard-square model, and the Widom–Rowlinson model. A weaker statement is proven for the Potts model. However, we will also give examples (see Section 3.6 below) in which the existence of an ordered state does not imply percolation of the corresponding ground state.

We emphasize that our setup is different from that in the Fortuin–Kasteleyn representation. Here the percolation clusters are defined directly in terms of the Gibbs state configuration. Therefore in general we do not expect to have direct relations between, e.g., correlation functions in the Gibbs state and corresponding percolation probabilities. Still we will see that for the model in Section 4 such a relation can in fact be established.

In the next section we present the general framework. This is implemented by the examples of Section 3 in the case of lattice systems. Section 4 is devoted to the continuum Widom–Rowlinson model.

2. GENERAL FRAMEWORK

We present the notation here in the case of lattice systems. The continuum model is contained in Section 4.

The Lattice. We restrict our attention to the d-dimensional lattice $\mathbb{Z}^d$, $d \geqslant 2$. This restriction is made for notational convenience and possible generalizations will be noted later. By $x \sim y$ we mean that x and $y \in \mathbb{Z}^d$ are nearest neighbors. Given $\Lambda \subset \mathbb{Z}^d$, $\partial\Lambda$ will denote the outer boundary of Λ, i.e., $\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda : y \sim x \text{ for some } y \in \Lambda\}$. In the sequel Λ is always a finite subset of $\mathbb{Z}^d$.

The Configuration Space. The single-site state space is denoted by S and it contains a finite number of elements ($|S| = q$). An infinite-volume configuration $\sigma = \{\sigma(x)\}_{x \in \mathbb{Z}^d}$ is an element of $\Omega = S^{\mathbb{Z}^d}$. A configuration σ is periodic if there is a vector $(k_1, \ldots, k_d) \in \mathbb{Z}^d$ with nonzero entries ($k_i \neq 0$, $i = 1, \ldots, d$) such that if $y = x + (n_1 k_1, \ldots, n_d k_d)$ for some $(n_1, \ldots, n_d) \in \mathbb{Z}^d$, then $\sigma(x) = \sigma(y)$.

The Hamiltonian and Its Ground States. The translation-invariant Hamiltonian H consists (for definiteness) of the nearest neighbor interaction $U: S \times S \to \mathbb{R}$ and the self-energy $V: S \to \mathbb{R}$

$$H(\sigma) = H(\sigma; J, h) = -J \sum_{x \sim y} U(\sigma(x), \sigma(y)) - h \sum_x V(\sigma(x)) \tag{2.1}$$

in which the infinite sums are formal. The real parameters $J \geqslant 0$ and h in (2.1) play the roles of a coupling coefficient and magnetic field (or chemical potential).

Consider $\sigma, \eta \in \Omega$ such that $\sigma(x) = \eta(x)$ for all $x \in \mathbb{Z}^d \backslash \Lambda$, for some finite $\Lambda \subset \mathbb{Z}^d$. Then σ and η are said to be *equal at infinity* or one is an *excitation* of the other. Their finite relative Hamiltonian $H(\sigma \mid \eta)$ is then

$$
\begin{aligned}
H(\sigma \mid \eta) = & -J \sum_{x \sim y} \left[U(\sigma(x), \sigma(y)) - U(\eta(x), \eta(y)) \right] \\
& -h \sum_{x} \left[V(\sigma(x)) - V(\eta(x)) \right]
\end{aligned}
\tag{2.2}
$$

in which the sum is now only over a finite number of terms. A configuration $\eta \in \Omega$ is a *ground state* if $H(\sigma \mid \eta) \geqslant 0$ for any excitation σ of η. The set of periodic ground-state configurations (PGSC) is denoted by $g(H)$. We always consider the case $|g(H)| > 1$.

The Gibbs Measure. Take $\eta \in g(H)$. The finite-volume (Λ) Gibbs static with η boundary conditions is the probability measure on the product space S^Λ defined by

$$
\mu^\eta_{\beta,\Lambda}(\sigma_\Lambda) = \frac{1}{Z_\Lambda(\beta, \eta)} \exp[-\beta H(\sigma^\eta_\Lambda \mid \eta)]
\tag{2.3}
$$

for $\sigma_\Lambda \in S^\Lambda$. In (2.3), $\beta > 0$ plays the role of the inverse temperature, $Z_\Lambda(\beta, \eta)$ is the normalization constant, and

$$
\sigma^\eta_\Lambda(x) = \begin{cases} \sigma_\Lambda(x) & \text{if } x \in \Lambda \\ \eta(x) & \text{if } x \in \Lambda^c = \mathbb{Z}^d \backslash \Lambda \end{cases}
\tag{2.4}
$$

Observe that σ^η_Λ is an excitation of η. The infinite-volume Gibbs measure μ^η_β is obtained by taking the limit as $\Lambda \nearrow \mathbb{Z}^d$ along suitable subsequences: μ^η_β is a measure on Ω endowed with the product sigma algebra (for details see Preston,[28] Georgii,[16] and Simon[31]). We remark that in general the limiting measure μ^η_β may depend on the subsequence; however, it is well known that this does not happen for a wide class of models at low temperature (in the realm of the Pirogov–Sinai theory) and in the cases we are going to consider we will show that μ^η_β does not depend on the subsequence. We thus restrict ourselves to this case. By phase coexistence at β we mean that $\mu^\eta_\beta \neq \mu^\sigma_\beta$ for some $\eta, \sigma \in g(H)$. In general we will say that μ_β is a Gibbs measure for the interaction U, V at the temperature β^{-1} if for all finite connected subsets Λ, $\mu_\beta(\sigma \mid \sigma(x) = \eta(x)$ for all $x \in \mathbb{Z}^d \backslash \Lambda) = \exp[-\beta H(\sigma^\eta_\Lambda \mid \eta)]/Z_\Lambda(\beta, \eta)$ for all σ_Λ, $\mu_\beta(d\eta)$-a.s.

Agreement with the PGSC. To understand the geometrical structure of such phases fix $\eta \in g(H)$ and the continuous map s_η: $\Omega \to \{0, 1\}^{\mathbb{Z}^d}$ (Ω and $\{0, 1\}^{\mathbb{Z}^d}$ are endowed with the local topology) expressing agreement or disagreement with the ground state η

$$(s_\eta(\sigma))(x) = \begin{cases} 1 & \text{if} \quad \sigma(x) = \eta(x) \\ 0 & \text{if} \quad \sigma(x) \neq \eta(x) \end{cases} \tag{2.5}$$

We denote by $\nu_\beta^\eta = \mu_\beta^\eta s_\eta^{-1}$ the measure on $\{0, 1\}^{\mathbb{Z}^d}$ induced by the map s_η.

Percolation of the Ground States. Percolation is defined in terms of regular site percolation for the measure ν_β^η: for $s \in \{0, 1\}^{\mathbb{Z}^d}$ define

$$C(s, \alpha) = \{x \in \mathbb{Z}^d: s(x) = \alpha\} \tag{2.6}$$

in which $\alpha = 0, 1$. Call $\bar{C}(s, \alpha)$ the maximal connected component in (2.6) containing the origin of $\mathbb{Z}^d$ $[\bar{C}(s, \alpha) = \varnothing$ if $s(0) \neq \alpha]$. The probability of η-percolation (respectively non-η-percolation) in the phase with boundary condition η is defined as

$$\theta_\beta^\eta(\alpha) = \nu_\beta^\eta(\{s: |\bar{C}(s, \alpha)| = \infty\}) \tag{2.7}$$

with $\alpha = 1$ (respectively $\alpha = 0$). We have η-percolation, or agreement percolation, (respectively non-η-percolation) if $\theta_\beta^\eta(1) > 0$ [respectively $\theta_\beta^\eta(0) > 0$]. Observe that not having η-percolation in general does not imply having non-η-percolation, and having η-percolation does not imply not having non-η-percolation.

Attractive Measures. If S is (partially or totally) ordered ($\geqslant$), Ω is partially ordered ($\succ$) by defining $\sigma' \succ \sigma$ if $\sigma'(x) \geqslant \sigma(x)$ for all $x \in \mathbb{Z}^d$. The order chosen for S may depend on the site x and, in the case of $S = \{0, 1\}$, we will call *standard order* the one induced by setting $0 \leqslant 1$ for all x. A measurable event A is said to be increasing if $\sigma \in A$ and $\sigma' \succ \sigma$ implies $\sigma' \in A$. A measure μ on Ω is attractive (FKG) if $\mu(A \cap B) \geqslant \mu(A)\,\mu(B)$ for all A, B increasing events.[14] Given two measures μ and μ', we say that μ dominates μ' if for every increasing event A, $\mu(A) \geqslant \mu'(A)$ (this is the so called FKG order of measures).

We will consider the validity of the following three statements:

(a) Phase coexistence implies percolation, i.e., given $\eta \in g(H)$, then

$$\mu_\beta^\eta \neq \mu_\beta^{\eta'} \quad \text{for some} \quad \eta' \in g(H) \Rightarrow \theta_\beta^\eta(1) > 0 \tag{2.8a}$$

(b) When $d = 2$, percolation implies coexistence,

$$\theta_\beta^\eta(1) > 0 \Leftrightarrow \mu_\beta^\eta \neq \mu_\beta^{\eta'} \qquad \text{for some} \quad \eta' \in g(H) \tag{2.8b}$$

(c) For $d=2$, regardless of the phase coexistence, there is no disagreement percolation,

$$\theta^{\eta}_{\beta}(0)=0 \tag{2.8c}$$

For the models we will be dealing with, if $d>2$, the statements (2.8b) and (2.8c) fail or they are expected to fail (see Aizenman *et al.*[1] and references therein). Statements (2.8a)–(2.8c) represent the best possible scenario, in which *phase coexistence* and *percolation* are equivalent. We stress once again that this scenario is not going to hold in general and we will produce counterexamples (even in $d=2$).

A tool in proving (2.8a) is the following:

Theorem 1 (Bricmont *et al.*[7]). In the notation of above, if there exists a bounded function $f: S \to \mathbb{R}$ and a constant δ such that

$$\int_{\Omega} f(\sigma(0))\,\mu^{\eta}_{\beta}(d\sigma) > \delta \tag{2.9}$$

while for all finite connected sets Λ containing the origin

$$\int_{\Omega} f(\sigma(0))\,\mu^{\eta}_{\beta}(d\sigma \mid s_{\eta}(x)=0 \text{ for all } x\in\partial\Lambda) \leqslant \delta \tag{2.10}$$

then $\theta^{\eta}_{\beta}(+1)\geqslant 0$, i.e., there is agreement percolation.

Proof. This is just Theorem 2 of Bricmont *et al.*[7] with a change of notation. ∎

In proving (2.8b) and (2.8c) we use the following theorem. Define $T_i\sigma(x)=\sigma(x-e_i)$, where e_i is the unit vector in the direction $i\in\{1,..., d\}$.

Theorem 2 (Gandolfi *et al.*[15]). If μ is a measure over $\{0,1\}^{\mathbb{Z}^d}$ which is attractive (with the standard order), invariant and ergodic under T_i, $i=1,2$, and invariant under reflection with respect to each axis, then if $\mu(\{s: |\bar{C}(s,\alpha)|=\infty\})>0$, then $\mu(\{s: |\bar{C}(s,1-\alpha)|=\infty\})=0$ for $\alpha=0,1$.

This theorem can be quite easily extended to cover more general cases than site percolation. In particular we will also use it in the absence of T_i invariance if the measure is T_i^2 invariant and ergodic ($i=1,2$). Furthermore, we will apply it to the case of bond percolation (see Section 3.4) and for continuous systems (Section 4). We will make the necessary remarks on the way.

3. LATTICE MODELS

As already mentioned in the introduction, the case in which $S=\{-1, +1\}$, $U(a, b,)=ab$, and $h=0$ (ferromagnetic Ising model) is well understood. In this case $g(H)=\{\eta_+, \eta_-\}$ $[\eta_+(x)=+1,\ x\in\mathbb{Z}^d$ and $\eta_-=-\eta_+]$ and for $\beta>\beta_c$ ($1/\beta_c$ is the critical temperature) there are two distinct translation-invariant extremal states obtained by taking η_+ and η_- as boundary conditions and the proof of the statements (2.8a)–(2.8c) can be found in Coniglio *et al.*[10]

3.1. Ising Antiferromagnet

$\mathbb{Z}^d$ is bipartite, that is, it can be split into two sublattices (in this case the points with even sum of coordinates $\mathbb{Z}^d_e$ and the ones with odd sum $\mathbb{Z}^d_o$) such that if $x, y\in\mathbb{Z}^d_e$ (or $x, y\in\mathbb{Z}^d_0$), then $x \not\sim y\cdot$ Fix $S=\{-1, +1\}$ and take

$$U(a, b)=-ab, \qquad V(a)=a \tag{3.1}$$

If $|h|<2dJ$, then $g(H)=\{\eta_e, \eta_o\}$, in which $\eta_e(x)=+1$ if $x\in\mathbb{Z}^d_e$, $\eta_e(x)=0$ otherwise, and $\eta_o=-\eta_e$ (see, for example, Dobrushin *et al.*[12]). The phase transition in this model has been studied by Dobrushin[11] and Heilmann.[19] Because of the bipartite structure, flipping the spins on the even (or odd) sites makes the model into a ferromagnetic Ising model with a staggered magnetic field, which is an FKG Gibbs model. In particular for magnetic field $h=0$, there is the usual Curie point T_c (the critical temperature for the ferromagnetic model above). In other words $\mu^{\eta_e}_\beta$ [as well as $\mu^{\eta_o}_\beta$ and any other Gibbs state with respect with the interaction (3.1)] is an FKG measure with respect to the order relation $\sigma\succ\sigma'$ iff $\sigma(x)\,\eta_e(x)\geqslant\sigma'(x)\,\eta_e(x)$ for all $x\in\mathbb{Z}^d$. A standard attractivity argument on the finite-volume Gibbs measures implies that $\mu^{\eta_e}_\beta$ and $\mu^{\eta_o}_\beta$ are well defined (independent of the way the infinite-volume limit is taken) and that if μ_β is a Gibbs measure with respect to the interaction (3.1), then $\mu^{\eta_o}_\beta\leqslant\mu_\beta\leqslant\mu^{\eta_e}_\beta$. For a full account of these arguments see for example, Section 9 of ref. 28.

Proposition 3.1. With the choices of (3.1) in (2.1), for any h and any $J\geqslant 0$, statements (2.8a)–(2.8c) hold.

Proof. We start by proving (2.8a). Take $\eta=\eta_e$. For any Λ containing the origin, we have that if $\mu^{\eta_e}_\beta\neq\mu^{\eta_o}_\beta$

$$\begin{aligned}&\nu^{\eta_e}_\beta(s(0)=1 \mid s(x)=0 \text{ for all } x\in\partial\Lambda)\\ &\quad=\mu^{\eta_e}_\beta(\sigma(0)=+1 \mid \sigma(x)=\eta_o(x), \text{ for all } x\in\partial\Lambda)\end{aligned}$$

$$
\begin{aligned}
&= \mu_\beta^{\eta_o}(\sigma(0) = +1 \mid \sigma(x) = \eta_o(x), \text{ for all } x \in \partial\Lambda) \\
&\leqslant \mu_\beta^{\eta_o}(\sigma(0) = +1) \\
&< \mu_\beta^{\eta_e}(\sigma(0) = +1) \\
&= \nu_\beta^{\eta_e}(s(0) = +1)
\end{aligned} \tag{3.2}
$$

The first equality is a change of notation, the second one is the Markov property, and the last one is again a change of notation. The first inequality follows from the attractivity of the measure $\mu_\beta^{\eta_e}$. The second one is a consequence of $\mu_\beta^{\eta_o} \leqslant \mu_\beta^{\eta_e}$, which implies that $\mu_\beta^{\eta_o}(\sigma(0) = +1) \leqslant \mu_\beta^{\eta_e}(\sigma(0) = +1)$ and of the fact that if $\mu_\beta^{\eta_o}(\sigma(0) = +1) = \mu_\beta^{\eta_e}(\sigma(0) = +1)$, then $\mu_\beta^{\eta_o} = \mu_\beta^{\eta_e}$ (see Theorem 1 in ref. 24), in contradiction with the assumption $\mu_\beta^{\eta_o} \neq \mu_\beta^{\eta_e}$. By Theorem 1 we have (2.8a).

In order to go on with the proof, let us make the following observations [in what follows, if a statement contains the configuration η, it means that it has to hold for each $\eta \in g(H)$]:

1. *μ_β^η is T_i^2 invariant.* This follows from the fact that μ_β^η is the monotone (by FKG) limit of a finite Gibbs state for the T_i^2-invariant boundary condition η and a T_i^2-invariant interaction (in fact T_i invariant).

2. $\mu_\beta^{\eta_e} = \mu_\beta^{\eta_o} T_i^{-1}$, as it follows easily form the definitions and the fact that the infinite-volume limit is well defined.

3. *μ_β^η is reflection invariant (with respect to each axis)* because the interaction and the boundary conditions are reflection invariant. Also ν_β^η is reflection invariant.

4. *μ_β^η is T_i^2 ergodic.* In fact, by the observations on the FKG properties of μ_β^η made before stating the proposition, it is extremal in the space of T_i^2-invariant Gibbs measure for the interaction (3.1) at the temperature β^{-1} and hence ergodic under T_i^2 ($i = 1, 2$).

5. *ν_β^η is T_i^2 invariant.* This follows from the fact that $T_i^{-2} s_\eta T_i^2 = s_\eta$ and observation 1, because

$$\nu_\beta^\eta = \mu_\beta^\eta s_\eta^{-1} = \mu_\beta^\eta T_i^{-2} s_\eta^{-1} T_i^2 = \mu_\beta^\eta s_\eta^{-1} T_i^2 \nu_\beta^\eta T_i^2$$

6. *ν_β^η is T_i^2 ergodic.* In fact, if $A \in \mathscr{B}(\{0, 1\}^{\mathbb{Z}^d})$ (is a measurable set) and $A = T_i^{-2} A$, then $s_\eta^{-1} A = T_i^2 s_\eta^{-1} T_i^{-2} A = T_i^2 s_\eta^{-1} A$, that is, $s_\eta^{-1} A$ is T_i^2 invariant. By observation 4 we conclude that $\nu_\beta^\eta(A) = 0$ or 1, that is, ergodicity.

7. *If $\mu_\beta^{\eta_e} = \mu_\beta^{\eta_o}$, then μ_β^η is T_i invariant.* This is a direct consequence of observation 2.
8. $\nu_\beta^{\eta_e} = \nu_\beta^{\eta_o} T_i^{-1}$. This follows from the fact that $s_{\eta_e} = T_i s_{\eta_o} T_i^{-1}$ and observation 2, because

$$\nu_\beta^{\eta_e} = \mu_\beta^{\eta_e} s_{\eta_e}^{-1} = \mu_\beta^{\eta_e} T_i s_{\eta_o}^{-1} T_i^{-1} = \mu_\beta^{\eta_o} s_{\eta_o}^{-1} T_i^{-1} = \nu_\beta^{\eta_o} T_i^{-1}$$

9. Define $S: \{0, 1\}^{\mathbb{Z}^d} \to \{0, 1\}^{\mathbb{Z}^d}$ as $(S(s))(x) = 1 - s(x)$, for all $s \in \{0, 1\}^{\mathbb{Z}^d}$ and all $x \in \mathbb{Z}^d$. If $\mu_\beta^{\eta_e} = \mu_\beta^{\eta_o}$, then $\nu_\beta^{\eta_e} S^{-1} = \nu_\beta^{\eta_o}$. This follows from the fact that $S s_{\eta_e} = s_{\eta_o}$, because

$$\nu_\beta^{\eta_e} S^{-1} = \mu_\beta^{\eta_e} s_{\eta_e}^{-1} S^{-1} = \mu_\beta^{\eta_o} s_{\eta_o}^{-1} = \nu_\beta^{\eta_o}$$

Let us now fix $d = 2$. By observations 3, 5, and 6, we see that ν_β^η meets the requirements for Theorem 2, but T_i is substituted with T_i^2. It is straightforward to modify its proof and extend the statement to the present case (as already observed by Klein and Yang[22]).

To prove (2.8b) it suffices to show that if $\mu_\beta^{\eta_e} = \mu_\beta^{\eta_o}$, then $\theta_\beta^{\eta_e}(1) = 0$. if $\theta_\beta^{\eta_e}(1) > 0$, then, by the modified version of Theorem 2, $\theta_\beta^{\eta_e}(0) = 0$. By observation 9, this implies that $\theta_\beta^{\eta_o}(1) = 0$, which is absurd by observation 8. Hence $\theta_\beta^{\eta_e}(1) = 0$. So (2.8b) is proven. Observe also that in this case not only $\theta_\beta^{\eta_e}(1) = 0$, but also $\theta_\beta^{\eta_e}(0) = 0$

In order to prove (2.8c), observe that we can restrict ourselves to the case in which $\mu_\eta^{\eta_e} \neq \mu_\beta^{\eta_o}$, because the other case has been considered above. In this case it is enough to recall that, by (2.8a), $\theta_\beta^\eta(1) > 0$, and hence, by the modified version of Theorem 2, $\theta_\beta^\eta(0) = 0$ and this concludes the proof. ∎

The results of Proposition 3.1 can be extended to more general bipartite lattices.

3.2. Hard-Core Lattices

The hard-core (or hard-squares) lattice model with activity λ is defined by the infinite-volume limit ($\Lambda \nearrow \mathbb{Z}^d$) of the measure $\mu_{\lambda,\Lambda}$ on the product space $\{0, 1\}^\Lambda$

$$\mu_{\lambda,\Lambda}^\gamma(\sigma_\Lambda) = \frac{\chi(\sigma_\Lambda^{\eta_\gamma})\, \lambda^{N_\Lambda}}{Z_\gamma(\lambda, \Lambda)} \tag{3.3}$$

In (3.3), $\gamma \epsilon \{e, o\}$ and $\eta_e(x) = 1$ if x is even, $\eta_e(x) = 0$ if x is odd [$\eta_o(x) = 1 - \eta_e(x)$ for all x]. further, $N_\Lambda = \sum_{x \in \Lambda} \sigma_\Lambda(x)$ and σ_Λ^η is defined in (2.4); $Z_\gamma(\gamma, \Lambda)$ is a normalization and for $\sigma \in \{0, 1\}^{\mathbb{Z}^d}$

$$\chi(\sigma) = \begin{cases} 1 & \text{if} \quad \sigma(x)\,\sigma(y) = 0 \quad \text{for all } x \sim y \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

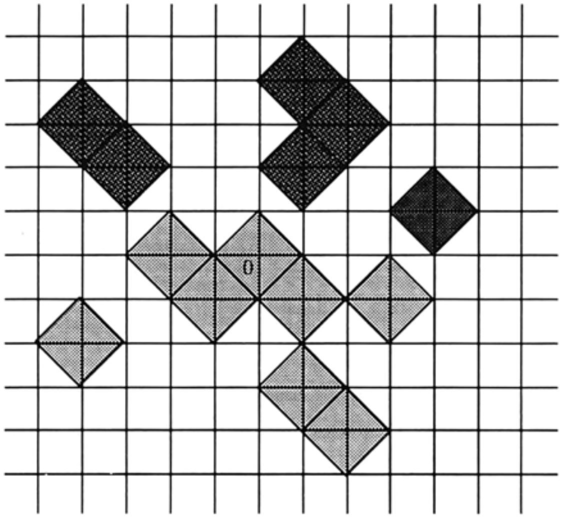

Fig. 1. A portion of a configuration of the hard-square lattice gas. The squares centered on odd sites are darker than the ones centered on even sites. In this figure there are three odd clusters and two even clusters.

Hence the model can be seen as a gas of hard (i.e., nonoverlapping) squares (see Fig. 1) or diamonds with fugacity λ. Call $C(\eta)=\bigcup_{x:\,\eta(x)=1} Q(x)$ $[Q(x)=\{y\in\mathbb{R}^d\colon \sum_{i=1}^d |y_i-x_i|=1\}]$ and denote by $C_0(\eta)$ the connected component of $C(\eta)$ that contains the origin (two squares touching at a corner are connected; see Fig. 1). There are clearly two different types of connected components of C: the ones for which the squares are centered on even sites and the ones for which they are centered on odd sites. We will call them type e and type o clusters, respectively. For $\gamma, \delta \in \{e, o\}$ define

$$\theta_\lambda^\gamma(\delta)=\mu_\lambda^\gamma(\{\eta\colon C_0(\eta) \text{ is of type } \delta \text{ and unbounded}\}) \tag{3.5}$$

(3.5) is clearly the analog of (2.7).

The hard-core model can be seen as a limit of the antiferromagnetic Ising model for $\beta\to\infty$ and $h\to 2dJ$ along $\beta(h-2dJ)=\operatorname{cotan}\theta$ $[\theta\in(0,\pi)]$. The phase diagram point $(2dJ,\ \beta=+\infty)$ is highly degenerate, since there are infinitely many (in general nonperiodic) ground states. In this limit, map $\sigma\in\{-1,+1\}^{\mathbb{Z}^d}\to\eta\in\{0,1\}^{\mathbb{Z}^d}$ by setting $\eta(x)=1$ if $\sigma(x)=-1$ and $\eta(x)=0$ if $\sigma(x)=+1$. We get a hard-square model with activity $\lambda=\exp(-2\operatorname{cotan}\theta)$ (see Dobrushin *et al.*[12] for details. This picture suggests that the critical fugacity (if it exists) should correspond to θ_c (see Fig. 2). We rephrase (2.8a)–(2.8c) into the following result.

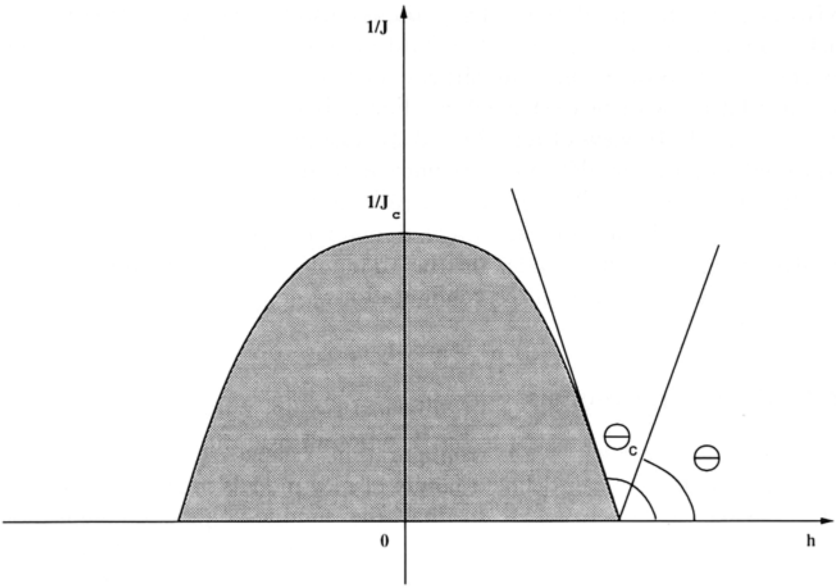

Fig. 2. The expected phase diagram of the antiferromagnetic Ising model for $d=2$. The shadowed area is the region of phase transition (this is only a qualitative diagram). The hard-square limit is obtained by taking $J \to \infty$ along the line with slope $\tan(\theta)$ shown on the figure. The phase transition for the hard-square model is expected to happen at $\theta = \theta_c$.

Proposition 3.2. For the hard-square model,

$$\mu_\lambda^e \neq \mu_\lambda^o \Rightarrow \theta_\lambda^e(e) > 0 \tag{3.6a}$$

Moreover, if $d=2$,

$$\theta_\lambda^e(e) > 0 \Leftrightarrow \mu_\lambda^e \neq \mu_\lambda^o \tag{3.6b}$$

and

$$\theta_\lambda^e(o) = 0 \tag{3.6c}$$

Proof. Define the partial order $\eta' \succ \eta$ if $\eta'(x) \geqslant \eta(x)$ for $x \in \mathbb{Z}_e^d$ and $\eta'(x) \leqslant \eta(x)$ if $x \in \mathbb{Z}_o^d$. It is easy to check that $\mu_{\lambda, \Lambda}$ and its infinite-volume limit are attractive. The proof is then the same of the one of Proposition 3.1. ∎

Remark. The motivation for Proposition 3.2 was provided by the work of Hu and Mak[20,21] in which a similar result is conjectured on the basis of computer simulations. They discuss also the case of hard-core particles on a triangular lattice, the hard-hexagon model.[20,21] Our result extends easily to other bipartite lattices, such as the hexagonal one. The triangular lattice with nearest neighbor bonds is not bipartite, so our proof does not work. In view of refs. 20 and 21, one expects that Proposition 3.2 still holds for this model, but it is unclear to us, especially in view of the results of the Section 3.4 on *many-layer* Ising models, whether Proposition 3.1 ($d=2$) holds for the whole domain of coexisting phases of the antiferromagnetic Ising model on the triangular lattice [in this case, for $h\in(0,6J)$, $g(H)$ contains three configurations].

3.3. Many-layer models

Given a model with configuration space $\{-1,+1\}^{\mathbb{Z}^d}$ and Hamiltonian H_1, we can define a family of new models indexed by integers $Q\geqslant 2$. Take $S=\{-1,+1\}^Q$ so that the infinite-volume configuration is of the form $\xi=((\sigma_1,\dots,\sigma_Q)\in S^{\mathbb{Z}^d}$ [to be identified with $\{-1,+1\}^{\mathbb{Z}^d}\times\cdots\times\{-1,+1\}^{\mathbb{Z}^d}$ (Q copies)]. Define the formal Hamiltonian as

$$H(\xi)=\sum_{i=1}^{Q} H_1(\sigma_i) \tag{3.7}$$

so that for boundary conditions $\omega=(\eta_1,\dots,\eta_Q)$, $\mu^{\omega}_{\beta,\Lambda}=\mu^{\eta_1}_{1,\beta,\Lambda}\times\cdots\times\mu^{\eta_Q}_{1,\beta,\Lambda}$ (where the subscript 1 refers to the system with Hamiltonian H_1) is the finite-volume Gibbs state with respect to H. Observe that $g(H)=(g(H_1))^Q$ (with the previous identification).

Duplicated Ising Model. Take $Q=2$ and H_1 as in (2.1), characterized by $U(a,b)=ab$ and $h=0$. As observed before, $g(H_1)=\{\eta_+,\eta_-\}$, and so $g(H)=\{\omega_{++},\omega_{+-},\omega_{-+},\omega_{--}\}$, where $\omega_{++}=(\eta_+,\eta_+)$ and so on.

Proposition 3.3. For the duplicated Ising model, statements (2.8a)–(2.8c) hold.

Proof. We consider only the case $\eta=\omega_{++}$ in (2.8). The other (three) cases are entirely similar. We want to estimate the expectation value of the sum of the spins at the origin given that $\xi(y)=(\sigma_1(y),\sigma_2(y))\neq(+1,+1)$ for $y\in\partial\Lambda$. In that case $\sigma_1(y)+\sigma_2(y)\leqslant 0$ for $y\in\partial\Lambda$, so that, with the notation $\mu(\sigma(0))=\int\sigma(0)\,\mu(d\sigma)$, we have

$$\mu_\beta^{\omega++}(\sigma_1(0)+\sigma_2(0) \mid s_{\omega_{++}}(\xi)=0 \text{ on } \partial\Lambda)$$
$$=\frac{1}{\mu_\beta^{\omega++}(s_{\omega_{++}}(\xi)=0 \text{ on } \partial\Lambda)} \sum_{\sigma_1', \sigma_2' \in \{-1,+1\}^{\partial\Lambda}} \{\chi(\sigma_1'+\sigma_2' \leqslant 0)$$
$$\times [\mu_{1,\beta}^{\eta+}(\sigma_1(0) \mid \sigma_1=\sigma_1' \text{ on } \partial\Lambda)+\mu_{1,\beta}^{\eta+}(\sigma_2(0) \mid \sigma_2=\sigma_2' \text{ on } \partial\Lambda)]$$
$$\times \mu_\beta^{\omega++}(\xi(x)=(\sigma_1'(x), \sigma_2'(x)), x \in \partial\Lambda)\} \leqslant 0 \tag{3.8}$$

as it follows from the fact that

$$[\mu_{1,\beta}^{\eta+}(\sigma_1(0) \mid \sigma_1=\sigma_1' \text{ on } \partial\Lambda)+\mu_{1,\beta}^{\eta+}(\sigma_2(0) \mid \sigma_2=\sigma_2' \text{ on } \partial\Lambda)]$$
$$\leqslant [\mu_{1,\beta}^{\eta+}(\sigma_1(0) \mid \sigma_1=\sigma_1' \text{ on } \partial\Lambda)+\mu_{1,\beta}^{\eta+}(\sigma_2(0) \mid \sigma_2=-\sigma_1' \text{ on } \partial\Lambda)]=0$$

in which the inequality follows from the FKG property and the last term vanishes by symmetry. On the other hand, if $\mu_{1,\beta}^{\eta+} \neq \mu_{1,\beta}^{\eta-}$,

$$\mu_\beta^{\omega++}(\sigma_1(0)+\sigma_2(0)) \equiv 2m^*(\beta)>0 \tag{3.9}$$

Apply Theorem 1 to get (2.8a) [note that for the case $\eta=\omega_{+-}$ one has to work in (3.8)–(3.9) with the differences $\sigma_1(0)-\sigma_2(0)$ instead]. With regard to (2.8b) and (2.8c), it is straightforward to see that $\nu_\beta^{\omega++}=\mu_\beta^{\omega++} s_{\omega_{++}}^{-1}$ is reflection invariant, ergodic under translations, and attractive. Hence we can apply Theorem 2 and get (2.8b) and (2.8c), as we did before. ∎

Remark. Note that as the phase transition is second order $[\mu_{1,\beta}^{\eta+}(\sigma(0)) \equiv m^*(\beta)$ is continuous at $\beta=\beta_c]$, the density of $(+1, +1)$ just below the critical temperature is only slightly above 1/4 and still the $(+, +)$ spins percolate in the $(+, +)$-state.[4] In the same way, the density of sites $x \in \mathbb{Z}^2$ where $(\sigma_1(x), \sigma_2(x)) \neq (+, +)$ is there only slightly below 3/4 and still, in the $(+, +)-$ state, they do not percolate.

The question therefore arises whether one can go arbitrarily far and construct examples where there is percolation for arbitrarily low densities or where there is no percolation no matter how large one makes the density. Such examples in fact exist (see, for example) Molchanov and Stepanov[26]), but they are rather singular. It may well be that a minimum density for having percolation actually exists for *good* Markov fields.

Conjecture. Given $\Omega=S^{\mathbb{Z}^d}$ $(|S|=q<\infty)$, there is a constant $c(d)>0$ (independent of q) such that for any translation-invariant pure Gibbs state μ on Ω for some translation- and rotation-invariant nearest neighbor interaction, if $\mu(\sigma(0)=a)<c(d)$, then $\{x: \sigma(x)=a\}$ does not percolate $(a \in S)$.

[4] The threshold for Bernoulli site percolation on $\mathbb{Z}^2$ is about 0.59.

In particular, we know of no such Gibbs state in two dimensions where one has percolation of a spin value with density less than 1/4. We believe, for example, that there is no $(+, +, +)$ percolation in the triplicated $+1$ state $\mu^+_{1,\beta} \times \mu^+_{1,\beta} \times \mu^+_{1,\beta}$ at temperatures close to the critical one. Obviously, in the conjecture, $c(d) \to 0$ as $d \to \infty$ [as does $p_c(d)$, the critical percolation probability for Bernoulli percolation].

A result in the direction of the conjecture is the following.

Proposition 3.4. Take H_1 of the ferromagnetic Ising model $[U(a, b) = ab]$ with $h = 0$. Take any β, including $\beta > \beta_c$, for which there is phase coexistence. There is $Q(\beta, d)$ such that for all $Q \geqslant Q(\beta, d)$

$$\theta^{\omega}_{\beta}(1) = 0 \tag{3.10}$$

for any $\omega \in g(H)$.

Proof. By direct computation

$$\max_{\xi'} \mu^{\omega}_{\beta}((s_{\omega}(\xi))(x) = 1 \mid \xi(y) = \xi'(y) \text{ for } y \sim x) = \left(\frac{1}{1 + \exp(-4d\beta J)}\right)^{Q} \tag{3.11}$$

We can now take Q sufficiently large so that

$$\left(\frac{1}{1 + \exp(-4d\beta J)}\right)^{Q} < p_c(d) \tag{3.12}$$

A standard domination argument concludes the proof {if $d = 2$, by using $p_c(2) > 1/2$ and $\beta_c = (1/2J) \log[1 + (\sqrt{2})]$, we get that there is a $\delta > 0$ such that if $Q = 24$, there is no percolation of the pattern ω if $\beta = \beta_c + \delta$}. ∎

Hence we have an example in which the measure is attractive, but nevertheless phase coexistence does not imply percolation. Of course Proposition 3.4 also holds for the *many-layer* version of other Markov fields.

Remark. If in $\mu^{\omega+-}_{\beta}$ there is with probability one some circuit $\partial\Lambda$ around the origin on which the two coordinates agree, i.e., $\sigma_1(y) = \sigma_2(y)$, $y \in \partial\Lambda$, then $\mu^{\eta+}_{1,\beta} = \mu^{\eta-}_{1,\beta}$ (the effect of the boundary will be clearly canceled by conditioning on the circuit $\partial\Lambda$). In other words, if $\mu^{\eta+}_{\beta} \neq \mu^{\eta-}_{\beta}$ (respectively, $\mu^{\eta_1}_{1,\beta} \neq \mu^{\eta_1}_{1,\beta}$ for $\eta_1, \eta_2 \in \Omega$), there must be *disagreement percolation* in $\mu^{\omega+-}_{\beta}$ (respectively in $\mu^{\eta_1}_{1,\beta} \times \mu^{\eta_2}_{1,\beta}$) in the sense of van den Berg[4] (see also van den Berg and Maes[6] for another coupling). Here *disagreement percolation* means that there is an infinite cluster on which $\sigma_1(x) \neq \sigma_2(x)$ [and does therefore not correspond to the notion introduced in (2.8c)].

This is applied in van den Berg and Steif[5] for the hard-core model of above. They take independently two realizations (σ_1, σ_2) according to the product coupling $\mu_\lambda^e \times \mu_\lambda^o$. A site $x \in \mathbb{Z}^2$ is a site of disagreement if $\sigma_1(x) \neq \sigma_2(x)$. They prove that $\mu_\lambda^e = \mu_\lambda^o$ if only if $\mu_\lambda^e \times \mu_\lambda^e(\{\sigma_1, \sigma_2)$ has an infinite path of disagreement$\}) = 0$. Using our general formulation, we make the following observation: if $\mu_\lambda^e \neq \mu_\lambda^o$, not only will we get disagreement percolation in the above sense, but in the state $\mu_\lambda^e \times \mu_\lambda^o$ this percolation will be over sites x where

$$\begin{aligned}(\sigma_1(x), \sigma_2(x)) &= (\eta_e(x), \eta_o(x)) \\ &= (1, 0) \qquad \text{for } x \text{ even} \\ &= (0, 1) \qquad \text{for } x \text{ odd}\end{aligned} \tag{3.13}$$

The reason is that $\mu_\lambda^e \neq \mu_\lambda^o$ is equivalent to $\mu^e \times \mu^o \neq \mu^o \times \mu^e$, implying the stability in $\mu_\lambda^e \times \mu_\lambda^o$ of the configuration (η_e, η_o) as given in (3.13).

3.4. The q-State Potts Model

In this case $S = \{1,..., q\}$. The Hamiltonian (2.1) is specified by taking $J > 0$, $h = 0$, and $U(a, b) = 0$ if $a = b$, $U(a, b) = -1$ if $a \neq b$.

It is straightforward to see that $g(H) = \{\eta_a : a \in S\}$, where $\eta_a(x) = a$ for all $x \in \mathbb{Z}^d$.

A very useful way to analyze the Potts model is to take the FK-representation of Fortuin and Kasteleyn.[13] For that we let $l(b) = 0, 1$ be a bond configuration. A bond $b = \langle xy \rangle$ is connecting nearest neighbors $x \sim y \in \mathrm{Z}^d$ and it can be *open* $[l(b) = 1]$ or *closed* $[l(b) = 0]$. We will say that $\langle x, y \rangle \cap \Lambda \neq \varnothing$ if $x \in \Lambda$ or $y \in \Lambda$ (or both). In $\Lambda \subset \mathbb{Z}^d$ we fix a bond configuration l by assigning to all bonds $b = \langle xy \rangle$ (connecting nearest neighbors $x \sim y$ at least one of which is inside Λ) the value 1 or 0. For bond percolation see the definitions in Grimmett[17] The event that 0 is connected to infinity through a chain of open bonds is denoted by $\{0 \leftrightarrow \infty\} \subset \{0, 1\}^{\mathbb{Z}^d}$. We define the following expectation for functions $f(\sigma_\Lambda)$ of the Potts model variables σ_Λ in the volume Λ with boundary conditions ξ:

$$\langle f \rangle_\Lambda^\xi(l) = \frac{1}{q^{n_\Lambda(l)}} \sum_{\sigma_\Lambda} f(\sigma_\Lambda) \prod_{\langle xy \rangle \cap \Lambda \neq \varnothing:\, l_{\langle x, y \rangle} = 1} \delta(\sigma_\Lambda^\xi(x), \sigma_\Lambda^\xi(y)) \tag{3.14}$$

Here δ is the Kronecker delta, and $n_\Lambda(l)$ is the number of connected l-clusters in the volume Λ so that the expectation (3.7) is normalized. The configuration σ_Λ^ξ is defined as in (2.4). The reason for introducing (3.7) is that the Potts model expectations in volume Λ with boundary conditions can be written as

$$\mu^{\xi}_{\beta,\Lambda}(f) = \frac{1}{Z_\Lambda(\beta,\xi)} \sum_{l} \prod_{b \cap \Lambda \neq \varnothing} p^{l_b}(1-p)^{1-l_b} q^{n_\Lambda(l)} \langle f \rangle^{\xi}_{\Lambda}(l)$$
$$\equiv \nu^{\mathrm{FK}}_{q,\beta,\Lambda}(\langle f \rangle^{\xi}_{\Lambda}(\cdot)) \tag{3.15}$$

when we put $p = 1 - e^{-\beta J}$. Here $\nu^{\mathrm{FK}}_{q,\beta,\Lambda}$ denotes the finite-volume Fortuin–Kasteleyn measure (or random cluster measure) on the bond configurations whose weights are defined by (3.15). Note that implicit in our definition is the boundary condition for the FK measure: using the terminology in Aizenman *et al.*,[2] $\nu^{\mathrm{FK}}_{q,\beta,\Lambda}$ is the finite-volume FK measure with *wired* boundary conditions. The infinite-volume limit of $\nu^{\mathrm{FK}}_{q,\beta,\Lambda}$ will be denoted by $\nu^{\mathrm{FK}}_{q,\beta}$.[2,17,18]

Proposition 3.5. For the Potts model, (2.8a) holds. Moreover, if $d = 2$ and $\mu^{\eta}_{\beta} \neq \mu^{\eta'}_{\beta}$ for $\eta, \eta' \in g(H)$, then

$$\theta^{\eta}_{\beta}(0) = 0 \tag{3.16}$$

and obviously also

$$\theta^{\eta'}_{\beta}(0) = 0$$

Proof. (2.8a) is easily proven by using the formula

$$\mu^{\eta_a}_{\beta}(\sigma(0) = a) = \frac{1}{q} + \left(\frac{q-1}{q}\right) \nu^{\mathrm{FK}}_{q,\beta}(\{0 \leftrightarrow \infty\}) \tag{3.17}$$

[see Theorem 2.3, formula (2.19), in Aizenman *et al.*[2]; see also Fortuin and Kasteleyn[13]] and that given the coupling between σ and l implicit in (3.14) and (3.15), $\sigma(x) = a$ if x belongs to one of the bonds in the infinite cluster of open bonds. By (3.17) the latter exists a.s. in the coexistence region, because there $\mu^{\eta_a}_{\beta}(\sigma(0) = a) > 1/q$ [Theorem 2.4, part (b), in Aizenman *et al.*[2]].

To prove (3.16), let us consider the FK measure $\nu^{\mathrm{FK}}_{q,\beta}$ associated to the extremal Potts measure $\mu^{\eta_a}_{\beta}$. The $\nu^{\mathrm{FK}}_{q,\beta}$ is translation invariant [Theorem 3.1, point (a), in Grimmett[18]] and ergodic under translations [Theorem 3.1, point (c), in Grimmett[17]]. Moreover, it is known that this measure is also attractive.[13,2,17,18] The reflection invariance is easily proven with the same argument used in the proof of Proposition 2.1 for the same property, because $\nu^{\mathrm{FK}}_{q,\beta}$ is FKG. A straightforward adaptation of the main theorem in Gandolfi *et al.*[15] to the case of bond percolation allows us to conclude that if there is percolation of open bonds, outside any box containing the origin there is a circuit of open bonds surrounding the origin (a.s.). Again by the coupling between σ and l, we have that if $\theta^{\eta_a}(1) > 0$, then $\mu_\beta^{\eta_a}$-a.s. every point is surrounded by a circuit on which $\sigma(x) = a$. By (2.8a) we conclude. ∎

Remark. Note that in $d = 2$ for $q > 4$ the "magnetization" $\mu^a(\{\sigma: \sigma(0) = a\})$ is believed to be discontinuous at $\beta = \beta_c$.[3] Therefore, in the two dimensional Potts model the lowest density of a ground-state configuration for which we know there is percolation is $1/4 + \varepsilon$ (for arbitrary $\varepsilon > 0$) and is obtained for $q = 4$ in the corresponding Gibbs state just below the critical point.[5]

3.5. Widom–Rowlinson Lattice Model

The statements (2.8a)–(2.8c) hold also for a class of models first introduced in Wheeler and Widom.[34] They are "spin-1" models with single-site state space $S = \{-1, 0, +1\}$ and Hamiltonian (2.1) determined by $U(a, b) = ab(1 - ab)$, $V(a) = a^2$, $0 \leq J < \infty$, and $h > 0$. This model was shown to have a phase transition by Lebowitz and Gallavotti (23) and to be attractive by Lebowitz and Monroe.[25]

The detailed analysis of these models follows along standard lines.

4. CONTINUUM MODEL

The continuum Widom–Rowlinson (WR) model[35] consists of particles of type A and type B having positions in $\mathbb{R}^d$ and fugacities z_A and z_B, respectively, whose interaction consists of the hard-core constraint that the centers of any two particles of different type must be at least distance R from each other. Cassandro *et al.*[9] show how this model can be obtained from a lattice model of the type described in Section 3.5.

More precisely, we take $\Lambda \subset \mathbb{R}^d$ a finite Borel set and let $x = (x_1, ..., x_{N_A})$ [respectively $y = (y_1, ..., y_{N_B}$, where N_A and N_B are positive integers, denote the position of particles A (respectively B), x_i, $y_i \in \Lambda$ for $i = 1, ..., N_A$ and $j = 1, ..., N_B$. Call X the space of a σ-finite integer-valued measures over Λ (and its Borel sets); our probability space will be $\Omega = X \times Y$ ($X = Y$ will have the topology of weak convergence, that characterizes its Borel sets). By separability, any element of X can be written as $\sum_{i \in I} \delta_{x_i}$ ($I \subset \mathbb{Z}$, $|I| < \infty$) and so we will use the notation $N_A(\omega)$, $N_B(\omega)$,

[5] This is similar to the case of duplicated Ising variables; see example 3.3.

$x(\omega)$, and $y(\omega)$ for $\omega \in \Omega$ with obvious meaning. The constraint that is imposed is determined by the hard-core length R:

$$\min_{i,j} |x_i - y_j| > R \tag{4.1}$$

Letting $I[x, y]$ denote the indicator function corresponding to (4.1) (I: $\Lambda^{N_A} \times \Lambda^{N_B} \to \{0, 1\}$), we put

$$Z^{\Lambda}_{N_A, N_B} = \int_{\Lambda^{N_A} \times \Lambda^{N_B}} d\lambda_{N_A}(x)\, d\lambda_{N_B}(y)\, I[x, y] \tag{4.2}$$

for $d\lambda_N(x) = d^d x_1 \cdots d^d x_N$ the N-product Lebesgue measure. Fixing the fugacities $z_A, z_B > 0$, we then have for the grand canonical partition function of the WR model

$$\Xi = \Xi(\Lambda, z_A, z_B) = \sum_{N_A, N_B} \frac{z_A^{N_A} z_B^{N_B}}{N_A!\, N_B!} Z^{\Lambda}_{N_A, N_B} \tag{4.3}$$

We will be mostly interested in the case $z_A = z_B = z$, for which we adopt the notation $\Xi(\Lambda, z)$.

So far we have not spoken of boundary conditions. Obviously we can fix the position of some particles by introducing extra constraints [beyond (4.1)]. For example, we speak of boundary conditions of type A if we replace $I[x, y]$ in (4.1) by $I_A[x, y] = I[x, y]\, I^A[y]$, where $I^A[y]$ is the indicator function corresponding to

$$\inf_{j,\, x \in \Lambda^c} |y_j - x| > R \tag{4.4}$$

Analogous definitions and notations apply for boundary conditions of type B. The grand canonical partition function is then changed into Ξ_γ, corresponding to the boundary conditions of type $\gamma = $ A, B.

The finite-volume Gibbs measure μ^γ_Λ for-boundary conditions $\gamma = $ A, B gives the probability of finding the particles in certain regions of Λ. If we condition on having N_A type A particles and N_B type B particles in Λ, the random field will have density

$$\frac{d\mu^\gamma_\Lambda(\cdot)|_{N_A(\omega) = N_A,\, N_B(\omega) = N_B}}{d\lambda_{N_A} \times d\lambda_{N_B}}(\omega) = \frac{1}{Z^{\Lambda}_{N_A, N_B}} \frac{z_A^{N_A} z_B^{N_B}}{N_A!\, N_B!} I_\gamma(\omega) \tag{4.5}$$

in which $I_\gamma(\omega) \equiv I_\gamma[x(\omega), y(\omega)](\omega \in \Omega)$ and analogously for $I(\omega)$; see (4.1). The infinite-volume measures are denoted by μ^γ and can be obtained as a limit from μ^γ_Λ as $\Lambda \nearrow \mathbb{R}^d$. The existence of such a limit as well as other properties of the limiting measures are proven by Cassandro *et al.*[9] and Lebowitz and Monroe.[25] The measure μ^γ does depend on the boundary condition γ if $z_A = z_B = z$ and z is sufficiently large.[29] Analogously, define μ as the infinite-volume limit of μ_Λ, defined as in (4.5) with $I_\gamma(\omega)$ replaced by $I(\omega)$.

Clusters and Percolation Probability. Define the function $Sp: \Lambda \times \{\omega: I(\omega) = 1\} \to \{\mathrm{A, B, W}\}$ as

$$Sp(x) = Sp(x, \omega) = \begin{cases} \mathrm{A} & \text{if } \operatorname{dist}(\mathbf{x}(\omega), \{x\}) < R/2 \\ \mathrm{B} & \text{if } \operatorname{dist}(\mathbf{y}(\omega), \{x\}) < R/2 \\ \mathrm{W} & \text{otherwise} \end{cases} \tag{4.6}$$

We can imagine the function Sp coloring the volume Λ in red (A), black (B), or white (W). From now on take $\gamma, \delta \in \{\mathrm{A, B}\}$. The γ cluster at the origin $[C_\gamma^0(\omega)]$ will then be defined as the connected component of $Sp(\cdot, \omega)^{-1}(\gamma)$ that contains the origin $[C_\gamma^0 = \varnothing$ if $Sp(0) \neq \gamma]$. The percolation probability is thus defined as

$$\theta^\gamma(\delta) = \mu^\gamma(\{\omega: \operatorname{diam}(C_\delta^0(\omega)) = \infty\}) \tag{4.7}$$

Proposition 4.1. Using the notations and definitions above with $z = z_A = z_B$, we have

$$\mu^{\mathrm{A}}(Sp(0) = \mathrm{A}) - \mu^{\mathrm{B}}(Sp(0) = \mathrm{A}) = \theta^{\mathrm{A}}(\mathrm{A}) - \theta^{\mathrm{A}}(\mathrm{B}) \tag{4.8}$$

implying

$$\theta^{\mathrm{A}}(\mathrm{A}) > \Leftarrow \mu^{\mathrm{A}} \neq \mu^{\mathrm{B}} \tag{4.9a}$$

In $d = 2$,

$$\theta^{\mathrm{A}}(\mathrm{B}) = 0 \tag{4.9b}$$

and

$$\theta^{\mathrm{A}}(\mathrm{A}) > 0 \Leftrightarrow \mu^{\mathrm{A}} \neq \mu^{\mathrm{B}} \tag{4.9c}$$

Remark 1. (4.8) says that the particle clusters in the WR model play a similar role to the random clusters in the FK representation of the Potts model. We believe this to be the first example where such a direct relation between the 'order parameter' and the cluster geometry is found. Note also that by attractivity[25] the left hand side of (4.8) is zero if and only if the A phase is different from the B phase.

Remark 2. We believe that the results in the proposition partially extend to the case where a hard-core condition is added between alike particles. The problem is that in that case the WR model loses the FKG property (defined for this model below) and, even if the core of the argument does not rely at all on attractivity, not having FKG would require extra technicalities related to the existence of infinitely volume-limits.

Remark 3. So far we have assumed for simplicity that the fugacities of the two types of particles are equal, $z_A = z_B = z$. Note, however, that if, says $z_A \geqslant z$, then $\mu^A_{z_A, z}$ stochastically dominates $\mu^A_{z, z}$, where we now explicitly indicate by subscripts the fugacities of A-type, respectively B-type, particles. This implies that if in $\mu^A_{z, z}$ there is percolation of A-type particles (as in the phase coexistence regime, $z > z_c$), then we get the same result for all A-particle fugacities $z_A \geqslant z$. Suppose we now integrate out the positions

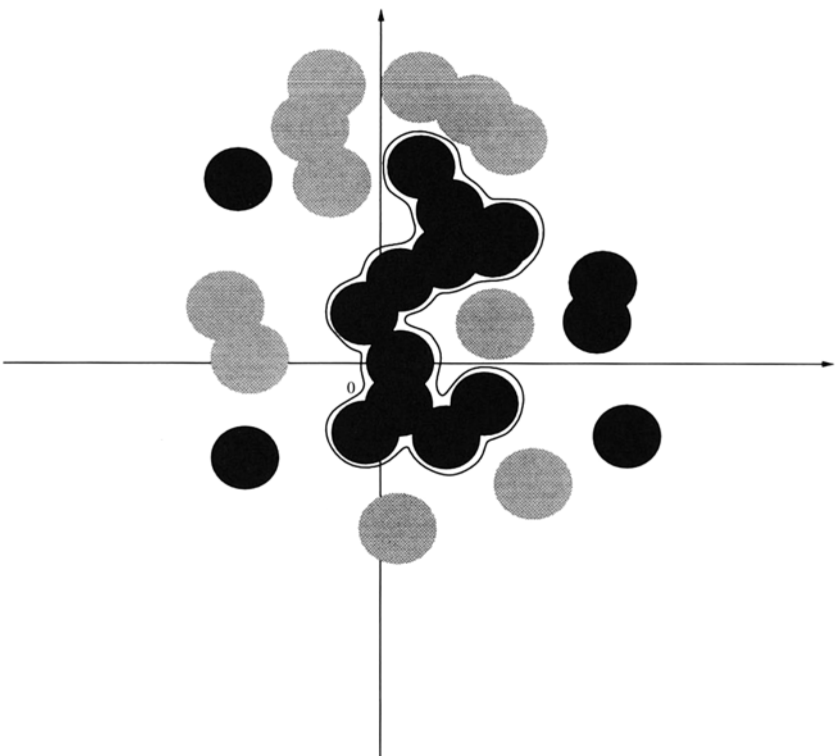

Fig. 3. A portion of a configuration of A (darker) and B (lighter) particles. There is an A cluster at the origin and it is separated from the other clusters by a white layer.

of the B particles. A simple calculation shows that we get a new measure for the A particles where now z (previously the fugacity of the B particles) plays the role of an inverse temperature. In that measure, for $z > z_c$, the A particles percolate for all values $z_A \geqslant z$.

Idea of the Proof. Consider an A cluster covering the origin. If it is bounded, then, by the hard-core constraint, there is necessarily a white region surrounding it; see Fig. 3. This effectively screens the origin from the external boundary condition. Therefore this contribution to the probability of having an A particle at the origin is the same in all states (i.e., regardless of the boundary conditions). What remains is the probability that the A cluster extends infinitely far. This idea can be most easily implemented through a discretization of the space.

Proof of Proposition 4.1. First we recall the result by Lebowitz and Monroe,[25] which says that μ is attractive with respect to the order $\succ$, $\omega \succ \omega'$ if $Sp(x, \omega) \geqslant Sp(x, \omega')$ for all $x \in \mathbb{R}^d$ (the order in $\{A\ \ B,\ \ W\}$ is $A \geqslant W \geqslant B$).

Take $\Lambda' \subset \Lambda \subset \mathbb{R}^d$ two spheres. Define the *finite-volume* percolation probability as $\theta^{\gamma}(\delta; \Lambda, \Lambda') = \mu^{\gamma}_{\Lambda}(C^0_{\delta}(\omega) \cap \partial\Lambda' \neq \varnothing)$ and by definition that

$$\lim_{\Lambda' \nearrow \mathbb{R}^d} \lim_{\Lambda \nearrow \mathbb{R}^d} \theta^{\gamma}(\delta; \Lambda, \Lambda') = \theta^{\gamma}(\delta) \tag{4.10}$$

By the FKG property of μ it is straightforward to see that the limits in (4.10) exist and that this limit can be computed in several other ways, for example, $\theta^{\gamma}(\delta) = \lim_{R \to \infty} \theta^{\gamma}(\delta; \Lambda(R), \Lambda(R/2))$, where $\Lambda(R)$ is the ball of radius R. Take $\varepsilon > 0$ small and cover $\mathbb{R}^d$ with a grid of spacing ε. Disregarding boundary problems, this naturally defines a partition of $\mathbb{R}^d$ into squares of sidelength ε (ε-squares). To control the errors made by the space discretization, we define

$$\begin{aligned} G(\varepsilon) = \{\omega\colon & \operatorname{dist}(\mathbf{x}, \mathbf{y}) > R + 6d\varepsilon, |\operatorname{dist}(\mathbf{x}, \partial\Lambda') - R| > 6d\varepsilon, \\ & |\operatorname{dist}(\mathbf{y}, \partial\Lambda') - R| > 6d\varepsilon, \\ & ||x_i - x_k| - R| > 6d\varepsilon, i \neq k \in \{1,..., N_A(\omega)\}\} \end{aligned} \tag{4.11}$$

A simple argument by contradiction yields the existence of $\Delta_{\Lambda}(\varepsilon) \geqslant 0$, vanishing as $\varepsilon \downarrow 0$ and such that

$$\mu^{\gamma}_{\Lambda}(G(\varepsilon)) > 1 - \Delta_{\Lambda}(\varepsilon) \tag{4.12}$$

with $\gamma \in \{A, B\}$.

The union of a finite number of ε-squares is an ε-cluster at the origin if the interior of this set is connected and if it contains the origin. Denote by $\mathscr{C}_\varepsilon$ the set of ε-clusters at the origin ($\varnothing \in \mathscr{C}_\varepsilon$).

We are now going to define a subset of Ω characterized by having a certain element of $\mathscr{C}_\varepsilon$ as minimal ε-covering of the corresponding maximal cluster at the origin. More precisely, given $C \in \mathscr{C}_\varepsilon$ define

$$\mathscr{A}_\varepsilon(C) = \{\omega\colon C^0_{\mathrm{A}}(\omega) \subset C \text{ and for all } C' \in \mathscr{C}_\varepsilon,\ C' \subsetneq C,\ C^0_{\mathrm{A}}(\omega) \not\subset C'\} \tag{4.13}$$

[Recall that $C^0_{\mathrm{A}}(\omega)$ is defined right before formula (4.7) and it denotes the original cluster at the origin, with no discretization.] We have then by construction that the probability to find an A particle at the origin is

$$\mu^{\mathrm{A}}_\Lambda(Sp(0) = \mathrm{A}) = \sum_{C \cap \partial\Lambda' = \varnothing} \mu^{\mathrm{A}}_\Lambda(\mathscr{A}_\varepsilon(C)) + \sum_{C \cap \partial\Lambda' \neq \varnothing} \mu^{\mathrm{A}}_\Lambda(\mathscr{A}_\varepsilon(C)) \tag{4.14}$$

where the sums are over $C \in \mathscr{C}(\varepsilon)$. We deal with the two terms in the right hand side of (4.14) separately.

First of all observe that by (4.12)

$$\begin{aligned}&\left|\sum_{C \cap \partial\Lambda' = \varnothing} \mu^{\mathrm{A}}_\Lambda(\mathscr{A}_\varepsilon(C)) - \sum_{C \cap \partial\Lambda' = \varnothing} \mu^{\mathrm{B}}_\Lambda(\mathscr{A}_\varepsilon(C))\right| \\ &\qquad \leqslant 2\Delta_\Lambda(\varepsilon) + \left|\sum_C (\mu^{\mathrm{A}}_\Lambda(\mathscr{A}_\varepsilon(C) \cap G(\varepsilon)) - \mu^{\mathrm{B}}_\Lambda(\mathscr{A}_\varepsilon(C) \cap G(\varepsilon)))\right|\end{aligned} \tag{4.15}$$

Define ∂C_{W} to be that subset of Ω such that $Sp(x, \omega) = \mathrm{W}$ for all x contained in an ε-square adjacent to the outer boundary of $C \in \mathscr{C}_\varepsilon$ (that is the external connected component of the boundary). Since $\mathscr{A}_\varepsilon(C) \cap G(\varepsilon) \subset \mathscr{A}_\varepsilon(C) \cap \partial C_{\mathrm{W}}$ we can continue (4.15), obtaining

$$\begin{aligned}&\left|\sum_{C \cap \partial\Lambda' = \varnothing} \mu^{\mathrm{A}}_\Lambda(\mathscr{A}_\varepsilon(C)) - \sum_{C \cap \partial\Lambda' = \varnothing} \mu^{\mathrm{B}}_\Lambda(\mathscr{A}_\varepsilon(C))\right| \\ &\qquad \leqslant 4\Delta_\Lambda(\varepsilon) + \sum_C |\mu^{\mathrm{A}}_\Lambda(\mathscr{A}_\varepsilon(C) \cap \partial C_{\mathrm{W}}) - \mu^{\mathrm{B}}_\Lambda(\mathscr{A}_\varepsilon(C) \cap \partial C_{\mathrm{W}})|\end{aligned} \tag{4.16}$$

By the Markov property of the μ^γ_Λ the sum in the right-hand side of (4.16) is zero. This is because by conditioning it is straightforward to see that every term in this sum is equal to

$$\begin{aligned}&\frac{\mu_\Lambda(\mathscr{A}_\varepsilon(C) \mid \partial C_{\mathrm{W}})\, \mu_\Lambda(I^{\mathrm{A}} \mid \partial C_{\mathrm{W}})\, \mu_\Lambda(\partial C_{\mathrm{W}})}{\mu_\Lambda(I^{\mathrm{A}})} \\ &\qquad - \frac{\mu_\Lambda(\mathscr{A}_\varepsilon(C) \mid \partial C_{\mathrm{W}})\, \mu_\Lambda(I^{\mathrm{B}} \mid \partial C_{\mathrm{W}})\, \mu_\Lambda(\partial C_{\mathrm{W}})}{\mu_\Lambda(I^{\mathrm{B}})} = 0\end{aligned} \tag{4.17}$$

where μ_Λ corresponds to the Gibbs distribution with free boundary conditions. The last equality follows because of the symmetry under exchange $\mathrm{A} \leftrightarrow \mathrm{B}$. Hence

$$\left| \sum_{C \cap \partial\Lambda' = \varnothing} \mu_\Lambda^{\mathrm{A}}(\mathscr{A}_\varepsilon(C)) - \sum_{C \cap \partial\Lambda' = \varnothing} \mu_\Lambda^{\mathrm{B}}(\mathscr{A}_\varepsilon(C)) \right| \leqslant 4\Delta_\Lambda(\varepsilon) \tag{4.18}$$

The absolute value of the difference between the second term in the right hand side of (4.14) and $\theta^{\mathrm{A}}(\mathrm{A}; \Lambda, \Lambda')$ is, by the definition (4.11) of $G(\varepsilon)$, smaller than $\Delta_\Lambda(\varepsilon)$. Combining this with (4.18) and writing (4.14) also for the measure μ_Λ^{B}, we get

$$|\mu_\Lambda^{\mathrm{A}}(Sp(0) = \mathrm{A}) - \mu_\Lambda^{\mathrm{B}}(Sp(0) = \mathrm{A}) + \theta^{\mathrm{B}}(\mathrm{A}; \Lambda, \Lambda') - \theta^{\mathrm{A}}(\mathrm{A}; \Lambda, \Lambda')| \leqslant 6\Delta_\Lambda(\varepsilon) \tag{4.19}$$

Hence the left-hand side of (4.19) is zero. The result (4.8) follows by taking the limits as indicated in (4.10). Condition (4.9a) follows from (4.8) and Remark 1.

In the case of two dimensions ($d = 2$), we use a generalized version of Theorem 2, extending the statement without difficulty to the continuum. This ensures that if $\theta^{\mathrm{A}}(\mathrm{A}) > 0$, then $\theta^{\mathrm{B}}(\mathrm{A}) = 0$ and then [by (4.8)] that $\mu^{\mathrm{A}} \neq \mu^{\mathrm{B}}$. If $\theta^{\mathrm{A}}(\mathrm{A}) = 0$, then the right-hand side of (4.8) is zero (by FKG) and so $\mu^{\mathrm{A}}(Sp(0) = \mathrm{A}) = \mu^{\mathrm{B}}(Sp(0) = \mathrm{A})$, which (as remarked above) implies that $\mu^{\mathrm{A}} = \mu^{\mathrm{B}}$. ∎

Note: After the completion of this work, we learnt that J. T. Chayes, L. Chayes, and R. Kotecký have obtained results similar to those of Section 4 of this paper [J. T. Chayes, L. Chayes, and R. Kotecký, The analysis of the Widom–Rowlinson model by stochastic geometric methods, Preprint (1994)].

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